A Control Barrier Function Approach to Constrained Resource Allocation Problems

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Overview

Constrained resource allocation problems in logistics, cloud computing, and UAV scheduling can often be modeled as variants of the **Facility Location Problem (FLP)**, which is **NP-hard even without constraints** due to its combinatorial and non-convex nature. In this abstraction:

- **Facility positions** encode resource attributes (e.g., server capacity, charging location),
- **Demand point positions** reflect task requirements (e.g., compute load, energy demand).

We extend the **Deterministic Annealing (DA)** framework, based on the **Maximum Entropy Principle (MEP)**, to incorporate capacity constraints, resulting in a nonlinear program with inequality constraints at each annealing step. To efficiently enforce feasibility and ensure convergence, we recast this problem as a control system:

- Introduce control-affine dynamics over decision variables,
- Use a control Lyapunov-like function and **Control Barrier Functions (CBFs)** for stability and constraint satisfaction,
- Solve a constrained quadratic program (QP) at each step.

Our method achieves up to **240× speedup** over classical solvers (e.g., SLSQP) with stronger constraint handling and scalability to dynamic settings.

Constrained Facility Location Problem

$$\min_{y_j\in\mathbb{R}^d,\ v_{j|i}\in\{0,1\}} \quad \mathscr{D} := \sum_{i=1}^{N} p_i \sum_{j=1}^{M} v_{j|i}\, d(x_i, y_j) \tag{1a}$$

$$\text{s.t.} \quad \sum_{j=1}^{M} v_{j|i} = 1 \quad \forall i = 1,\dots,N \tag{1b}$$

$$L_j \le \sum_{i=1}^{N} p_i v_{j|i} c_{ij} \le C_j \quad \forall j = 1,\dots,M \tag{1c}$$

Definitions: $v_{j|i} \in \{0,1\}$: assignment of demand i to facility j; $y_j, x_i \in \mathbb{R}^d$: facility and demand locations; $p_i > 0$: demand weights with $\sum_i p_i = 1$; $d(x_i, y_j)$: assignment cost (e.g., squared distance); c_{ij}: resource usage of demand i at facility j; L_j, C_j: lower and upper capacity bounds.

MEP-based Formulation for FLP

• Unconstrained DA for FLP [1] – $\mathbf{P}_{\mathbf{unconstr}}(\beta)$

We relax binary assignments $\{v_{j|i}\}$ to invoke **MEP**, following the **Deterministic Annealing (DA)** approach introduced in [1] formulation introduces soft assignment probabilities $p^{\beta}_{j|i} \in [0,1]$ and minimizes the β-parameterized free energy:

$$\min_{y_j^{\beta}\in\mathbb{R}^d,\ p^{\beta}_{j|i}\in[0,1]} \quad \mathscr{F}^{\beta} := \sum_{i=1}^{N} p_i \sum_{j=1}^{M} p^{\beta}_{j|i}\, d(x_i, y_j^{\beta}) + \frac{1}{\beta}\sum_{i=1}^{N} p_i \sum_{j=1}^{M} p^{\beta}_{j|i} \log p^{\beta}_{j|i} \tag{2a}$$

$$\text{s.t.} \quad \sum_{j=1}^{M} p^{\beta}_{j|i} = 1, \quad \forall i = 1,\dots,N \tag{2b}$$

Solution (fixed-point update):

$$p^{\beta}_{j|i} = \frac{e^{-\beta d(x_i, y_j^{\beta})}}{\sum_{\ell=1}^{M} e^{-\beta d(x_i, y_\ell^{\beta})}}, \qquad y_j^{\beta} = \frac{\sum_{i=1}^{N} p_i\, p^{\beta}_{j|i}\, x_i}{\sum_{i=1}^{N} p_i\, p^{\beta}_{j|i}}$$

These updates are iterated at increasing values of β, starting from a low-entropy initialization (e.g., uniform assignments), gradually converging to localized, near-binary assignments as $\beta \to \infty$.

• Constrained DA for FLP – $\mathbf{P}_{\mathbf{constr}}(\beta)$

Same as (2), with the addition of capacity constraints:

$$L_j \le \sum_{i=1}^{N} p_i\, p^{\beta}_{j|i}\, c_{ij} \le C_j, \quad \forall j = 1,\dots,M \tag{3}$$

Control-Based Optimization Framework

Theorem. Consider the relaxed constrained problem $\mathrm{P}_{\mathrm{constr}}(\beta)$, with cost shifted to ensure non-negativity as $\tilde{\mathscr{F}}^{\beta} := \mathscr{F}^{\beta} + \frac{\log M}{\beta}$. Define control-affine dynamics:

$$\dot{p}_{j|i} = v_{ij}, \qquad \dot{y}_j = u_j,$$

initialized with a feasible $p_{j|i}(0) \in (0,1)$ and at least one resource strictly within its capacity bounds. Let (v_{ij}, u_j, δ) solve the quadratic program:

$$\begin{aligned} \min \quad & \sum_{i,j} v_{ij}^2 + \sum_j \|u_j\|^2 + q\delta^2 \\ \text{s.t.} \quad & \dot{\tilde{\mathscr{F}}} < -\mu\tilde{\mathscr{F}} + \delta \\ & \dot{\phi}_i = 0 \quad \forall i \\ & \dot{\psi}_{c,j} \ge -\alpha_c \psi_{c,j}, \quad \dot{\psi}_{l,j} \ge -\alpha_l \psi_{l,j} \quad \forall j \\ & \dot{\xi}_{j|i} \ge -\alpha_\xi \xi_{j|i} \quad \forall i,j \end{aligned}$$

Constraint definitions:

- $\phi_i := \sum_j p_{j|i} - 1$: ensures valid assignments,
- $\psi_{c,j} := C_j - \sum_i p_i p_{j|i} c_{ij}$: upper capacity constraint,
- $\psi_{l,j} := \sum_i p_i p_{j|i} c_{ij} - L_j$: lower capacity constraint,
- $\xi_{j|i} := p_{j|i}(1 - p_{j|i})$: enforces interiority.

Constants $q, \mu, \alpha_c, \alpha_l, \alpha_\xi > 0$ control convergence and constraint enforcement.

Conclusion. The system trajectories $\{p_{j|i}(t), y_j(t)\}$ remain feasible and converge to a KKT point of $\mathrm{P}_{\mathrm{constr}}(\beta)$ as $t \to \infty$.

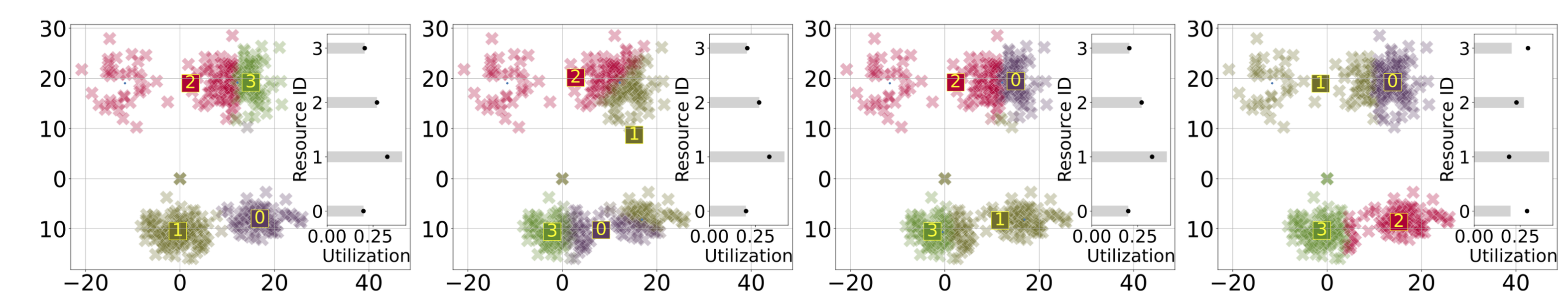
Remark 1. At convergence, $\{p_{j|i}\}$ defines a soft assignment of demand to resources, and each facility location y_j lies at the weighted centroid of its assigned demands.

Remark 2. Our result generalizes to a broad class of nonlinear optimization problems, provided the following mild regularity conditions hold:

- **Linear independence** of active constraint gradients along the trajectory,
- **Lipschitz continuity** of the objective and constraint gradients along the trajectory,
- **Coercivity** of the objective over the feasible set.

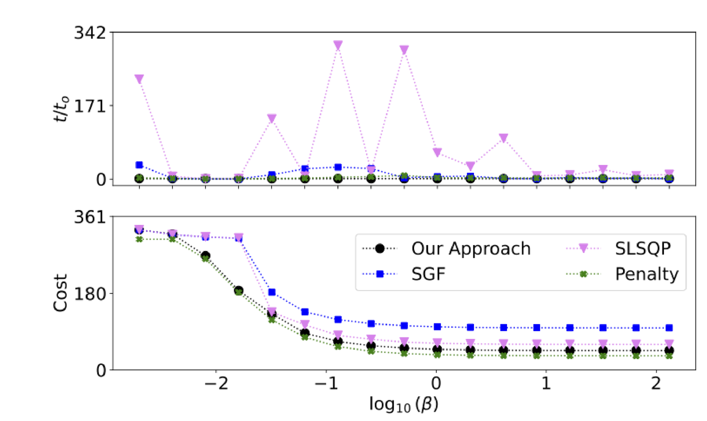
The capacitated facility location problem serves as a concrete instance of this general control-theoretic framework.

Simulation Results



(a) Our Approach (b) SGF (c) SLSQP (d) DA-P

Figure 1. The figure shows a capacitated FLP with 400 demand points in 4 clusters, solved using the four methods. Final resource utilization is shown to the right of each subplot. All the approaches maintain feasibility except the DA penalty-based method. Runtimes (in sec): $\{46, 210, 1600, 60\}$. Final costs: $\{46, 99, 60, 33\}$ units.

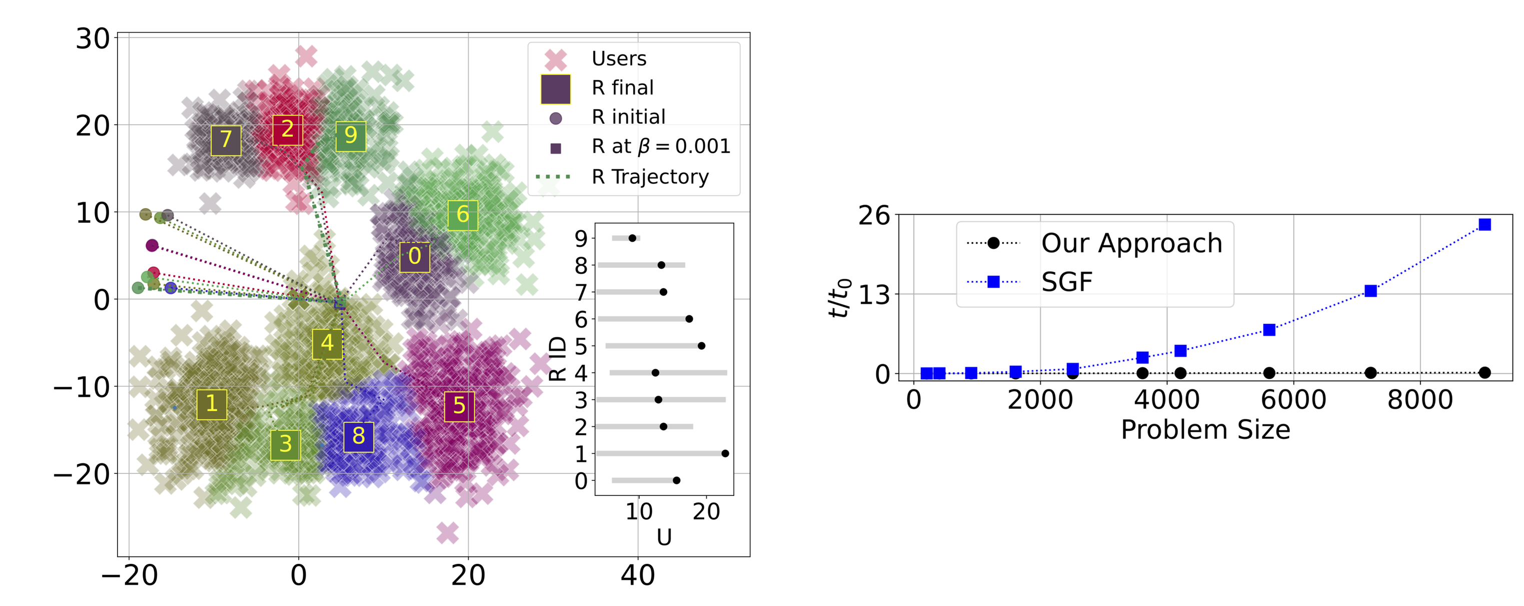


(a) β grows from $10^{-3} \to 10^{2}$

β	P-Size	Our Method		SGF		SLSQP	
		Cost	Time	Cost	Time	Cost	Time
10^{-3}	204	37.6	0.3	36.3	0.8	38.1	0.5
	906	51.8	0.7	52.0	11.3	51.9	35.8
	1608	63.2	1.5	63.2	43.2	63.2	271.5
	3010	118.5	3.4	118.4	163.8	118.3	2205.5
1	204	10.8	0.4	11.7	2.2	10.8	0.3
	906	12.5	1.1	23.3	14.8	12.3	66.0
	1608	10.3	1.5	22.9	51.8	10.5	356.7
	3010	11.5	9.9	24.0	170.8	11.0	5670.1
100	204	10.3	0.3	10.9	3.8	10.3	0.3
	906	10.9	1.0	21.6	15.7	10.9	11.0
	1608	12.0	1.3	20.6	50.3	11.5	364.0
	3010	13.7	4.2	27.5	185.1	13.0	4570.7

(b) β remains fixed.

Figure 2. Time and cost comparison for different methods.



(a) Solution using our approach (b) QP runtime comparison

Figure 3. Capacitated FLP solution using our CBF-based approach for $N = 1000$, $M = 10$. The cluster split of users: $[0.11, 0.07, 0.11, 0.09, 0.13, 0.14, 0.02, 0.11, 0.14, 0.08]$ and facility (R) utilization (U) constraints are shown at the bottom right. The figure also shows splitting of facilities into distinct clusters as $\beta \in [10^{-3}, 100]$ is increased during annealing. A problem of this size is not solvable by SGF approach.

References

[1] Kenneth Rose. Deterministic annealing for clustering, compression, classification, regression, and related optimization problems. *Proceedings of the IEEE*, 86(11):2210–2239, 1998.

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